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SUPERCRITICAL FLOWS FROM BENEATH A SHIELD

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The theory of motion of an ideal incompressible heavy liquid with free boundaries is a unique branch of classical hydrodynamics. Interest in such flows exists first, because of their great practical importance and, second, because of the richness, uniqueness, and difficulty in mathematical description of the problems which develop. Many studies have been published concerning precise solutions of the steady-state equations of motion of a vortex-free liquid with free boundaries. Proof of the existence of traveling waves was first offered by Nekrasov in 1921, then independently by Levi-Civita in 1925 for an infinitely deep liquid. Later, Struik, followed by Nekrasov, established analogous theorems for liquids of finite depth. It was assumed in their studies that the flow was subcritical, i.e., that the velocity of the main flow was less than the propagation velocity of infinitely low amplitude waves. In the 1950s a number of studies appeared on steady-state supercritical flows. Works by Zherbe, Moiseev, and Ter-Krikorov proved the existence of supercritical waves above a rough periodic bottom. Information on these and other studies on the same theme can be found in [1, 2]. In 1982, the existence of subcritical flows about a rough aperiodic bottom was established [3]. The existence of combined waves was first strictly proven by Lavrent'ev [4] in 1946 using the variation principles of conformal and quasiconformal transform theory which he developed. Another method of proof was proposed in [5]. Both proofs are based on principles of nonlinear shallow wave theory and show that this theory can be used for asymptotic representation of precise solutions of the combined wave problem. If we admit the possibility of contact of the free surface with rigid boundaries, the corresponding nonlinear boundary conditions become much more complicated. The present study will examine the two-dimensional problem of flow of a heavy vortex-free ideal liquid from underneath a planar horizontal lid over a smooth horizontal bottom. The flow is assumed supercritical: $U > \sqrt{gh_0}$, although the characteristic flow velocity U is assumed to differ little from the critical velocity $\sqrt{gh_0}$ (where g is the acceleration of gravity and h_0 is the characteristic liquid thickness). We will find the approximate form of the free surface and present a method for proving the existence of flows that are not uniform. It is thus just this fact which justifies the approximate solution. The problem to be formulated herein differs greatly from that of the combined wave. Nevertheless, the method of study to be presented below has much in common with that proposed by Friedrichs and Heyers [5].

1. Formulation of the Problem. To describe the liquid flow we choose as independent variables [1] the dimensionless velocity potential φ and flow function ψ. Such a choice of variables allows us to operate in a fixed belt between two flow lines ψ = const rather than in a partially unknown flow region. As is well known, the complex velocity potential $\chi = \varphi + i\psi$ is an analytical function of the variable $z = x + iy$. The conjugate complex velocity $\bar{w} = d\chi/dz$ is also an analytical function of z. After the substitution $w = \exp\{-i(\theta + i\tau\}$ the problem of liquid flow is reduced [1] to search for an analytic function $\theta + i\tau$ of

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the variable χ in a unit horizontal band, with boundary condition $\theta_\psi - \lambda \exp\{-3\tau\} \sin\theta = 0$ on the "free" surface $\psi = 1$, $\varphi > 0$ with constant $\lambda = gh_0U^{-2}$ (without loss of generality, we assume that the point of contact of the free surface and lid transforms to a point $\varphi = 0$, $\psi = 1$). Since on the bottom and lid the angle of inclination of the flow velocity should coincide with the angle of inclination of the tangent, we have $\theta = 0$ at $\psi = 0$ and $\psi = 1$, $\varphi \le 1$.

It is assumed that $\lambda = 1 - \varepsilon^2\mu_0{}^2/3$ (ε is a small parameter, $\mu_0 > 0$). In other words, the basic flow is assumed supercritical, but differing little from critical. It will be desirable to make the replacement of variables used in shallow-water theory $x = \varepsilon\varphi$, $y = \psi$ and take

$$F(u, v, \varepsilon) = \varepsilon^{-5}\left(1 - \varepsilon^2\mu_0^2/3\right)\left(\exp\{-3\varepsilon^2 v\}\sin\varepsilon^3 u - \varepsilon^3 u\right) = -3uv + \varepsilon^2 F_1(u, v, \varepsilon) \tag{1.1}$$

with the function

$$F_1(u, v, \varepsilon) = 3\mu_0^2 uv + u\left(\exp\{-3\varepsilon^2 v\} + 3\varepsilon^2 v - 1\right) + \varepsilon^{-7}\left(1 - \varepsilon^2\mu_0^2/3\right)\exp\{-3\varepsilon^2 v\}\left(\sin\varepsilon^3 u - \varepsilon^3 u\right).$$

In the new notation the original problem is reformulated as one of finding in a plane $-\infty < x < \infty$, $0 < y < 1$ pairs of functions (u, v) from the system

$$u_y + v_x = 0,\ \varepsilon^2 u_x - v_y = 0 \tag{1.2}$$

with boundary conditions on the "solid walls"

$$u = 0\ (y = 0);\ u = 0\ (y = 1,\ x \leqslant 0) \tag{1.3}$$

and condition on the "free surface"

$$u_y - \left(1 - \varepsilon^2\mu_0^2/3\right)u = \varepsilon^2 F(u, v, \varepsilon) \quad (y = 1,\ x > 0). \tag{1.4}$$

Since the function v "conjugate" to u can be found only to the accuracy of an arbitrary constant, its unique determination requires an additional condition. We assume that

$$v(x,\ y) \to 0\ (x \to \infty). \tag{1.5}$$

With the given function $\varphi(x) = u(x, 1)$, from system (1.2) with boundary condition $u(x, 0) = 0$ and condition (1.5) at infinity the functions $u(x, y)$ and $v(x, y)$ can be uniquely reconstructed. Therefore, for further solution of the problem of Eqs. (1.2)-(1.5) we will term the function $\varphi(x)$ [or the pair $\varphi(x)$, $\psi(x) = v(x, 1)$].

2. Approximate Solution. In [6] the boundary problem of Eqs. (1.2) and (1.3) was studied with boundary condition

$$u_y - \left(1 - \varepsilon^2\mu_0^2/3\right)u = \varepsilon^2 f(x, \varepsilon) \quad (y = 1,\ x > 0) \tag{2.1}$$

and for the functions $\varphi(x) = u(x, 1)$, $\psi(x) = v(x, 1)$ in the range $0 < \alpha \le 1/2$ the forms

$$\begin{gathered}\varphi(x) = Af(x,\ 0) + A(f(x,\ \varepsilon) - f(x,\ 0)) + \varepsilon^{1-\alpha}N(\varepsilon)f(x,\ \varepsilon),\\ \psi(x) = -D^{-1}\varphi(x) + \varepsilon^{1-\alpha}K(\varepsilon)\varphi(x)\end{gathered} \tag{2.2}$$

were found. These expressions consider the condition (1.5) at infinity and use the notation

$$D^{-1}\varphi(x) = -\int\limits_x^\infty \varphi(x)\,dx.$$

The function $w = Ah$ at $x > 0$ is a solution of the boundary problem

$$-w''(x) + \mu_0^2 w(x) = 3h(x), \quad w(0) = w(\infty) = 0 \tag{2.3}$$

and vanishes at $x \le 0$. The integral operators N and K act in the induced spaces of the functions of the problem of Eqs. (1.2), (1.3), and (2.1), which will be defined below.

The set of functions exponentially decaying with exponent $\rho \ge 0$ defined on the entire R axis and having a finite Holder C^α-norm ($0 \le \alpha \le 1$) will be denoted by $E_\alpha(\rho)$, and the norm will be introduced by the equation

$$\|u(x)\|_{E_\alpha(\rho)} = \|u(x)\exp\{\rho|x|\}\|_{C^\alpha(R)}, \tag{2.4}$$

$E_\alpha^+(\rho)$ is the set of Holder functions exponentially decaying on the positive semiaxis R^+, for which norm (2.4) is finite with R^+ in place of R, $\mathring{E}_\alpha^+(\rho)$ is the set of functions from $E_\alpha^+(\rho)$ equal to zero at x = 0. It is clear that $\mathring{E}_\alpha^+(\rho)$ can be identified with the subspace of the functions from $E_\alpha(\rho)$ equal to zero at $x \le 0$.

It was shown in [6] that for $0 < \rho < \mu_0$, $0 < \alpha \le 1/2$ and sufficiently small $0 < \varepsilon < \varepsilon_0(\mu_0, \rho)$, the operator $N(\varepsilon)$ acts from the space $E_\alpha^+(\rho)$ into the space $\mathring{E}_\alpha^+(\rho)$ and its norm over ε is limited by a constant dependent on μ_0 and ρ only. The singular integral operator at $0 < \alpha < 1$ acts at these ε and $\rho < \pi/\varepsilon - \delta$ from the space $E_\alpha(\rho)$ into itself. Its norm is uniformly limited by a constant dependent only on $\delta > 0$, α, and ε_0. Therefore, below we will assume that $0 < \alpha \le 1/2$, $0 < \rho < \mu_0$, $0 < \varepsilon < \varepsilon_0$ and the number ε_0 will be chosen such that these properties of the operators $N(\varepsilon)$ and $K(\varepsilon)$ will be satisfied.

The operator A is independent of ε. It acts from the space $E_\alpha^+(\rho)$ at $0 < \rho < \mu_0$ into the space $\mathring{E}_\alpha^+(\rho)$ and is finite. Its norm depends on μ_0 and ρ.

If we denote on the spaces introduced by V, then we can easily verify the estimate of the product $\|uv\|_V \le C(\rho)\|u\|_V\|v\|_V$, which permits estimation of the norms of the entire analytical complex functions of the type of Eq. (1.1) in the spaces introduced. More precisely. if $F(0) = 0$ and the series $\sum\limits_{|\alpha|>0} A_\alpha t^\alpha = F(t)$ converges for all $t = (t_1, \ldots, t_m)$, then for the complex function

$$\|F(v)\|_V \leqslant \sum_{|\alpha|>0} C^{|\alpha|}(\rho)|A_\alpha|X^\alpha, \quad X = (\|v_1\|_V, \ldots, \|v_m\|_V). \tag{2.5}$$

Further, let φ be a solution of Eqs. (1.2)-(1.5) and $\varphi(x) = u(x, 1) \in \mathring{E}_\alpha^+(\rho)$. On the basis of the above, there follows from Eq. (2.2) the equality $v(x, 1) = -D^{-1}\varphi(x) + o(\varepsilon)$ and, therefore, for y = 1, $F(u, v, \varepsilon) = 3\varphi D^{-1}\varphi + o(\varepsilon)$. Again, from Eq. (2.2), $\varphi(x) = 3A\cdot(\varphi D^{-1}\varphi) + o(\varepsilon)$. It is clear that the approximate solution $\varphi_0(x)$ of Eqs. (1.2)-(1.5) must be found from the integral nonlinear equation $\varphi_0(x) = 3A(\varphi_0 D^{-1}\varphi_0)$ or the equivalent boundary problem of Eq. (2.3)

$$-\varphi_0'' + \mu_0^2\varphi_0 = 3\varphi_0 D^{-1}\varphi_0, \quad \varphi_0'(0) = \varphi_0(\infty) = 0. \tag{2.6}$$

This equation coincides with the equation of combined waves. Its solution is well known:

$$\varphi_0(x) = -\mu_0^3 \operatorname{sh}\frac{\mu_0 x}{2}\Big/\operatorname{ch}^3\frac{\mu_0 x}{2}, \tag{2.7}$$

and can be found from Eq. (2.6) after the substitution $D^{-1}\varphi_0 = w$.

<u>3. Exact Solution.</u> The problem of Eqs. (1.2)-(1.5) differs greatly from the combined wave problem. Its rectilinear expansion in powers of ε, used in the narrow-band method to construct an asymptotic solution, will not function here, since the solution will undoubtedly not be smooth at the point x = 0, y = 1. Therefore, to justify approximate solution (2.6) it is necessary to prove the existence of a precise solution having the required properties. It can easily be seen that the function φ_0, extended onto the negative semiaxis is piecewise-smooth and belongs to the space $\mathring{E}_{1/2}^+(\mu_0)$. Therefore, the number α figuring in the considerations of the preceding section, will be taken equal to 1/2 below.

With consideration of Eq. (2.2), the solution φ, ψ of Eqs. (1.2)-(1.5) has the form

$$\varphi(x) = \varphi_0(x) + \varepsilon^{1/2}\eta(x), \quad \psi(x) = -D^{-1}\varphi(x) + \varepsilon^{1/2}K(\varepsilon)\varphi(x).$$

If these expressions are substituted in boundary conditions (1.4) and Eq. (2.2) is used, then the problem of Eqs. (1.2)-(1.5) is reduced to search for a solution $\eta \in \mathring{E}_{1/2}^+(\rho)$, $\rho < \mu_0$ of the nonlinear integral equation

$$\eta = 3A(\eta D^{-1}\varphi_0 + \varphi_0 D^{-1}\eta) + F_2(\varphi_0, \varepsilon) + \varepsilon^{1/2}F_3(\eta, \varphi_0, \varepsilon), \tag{2.8}$$

where $F_2(\varphi_0, \varepsilon) = 3A(-\varphi_0 K(\varepsilon)\varphi_0 + N(\varepsilon)(\varphi_0 D^{-1}\varphi_0))$;

$$F_3(\varphi_0, \varepsilon) = 3A(\eta D^{-1}\eta - \eta K(\varepsilon)\varphi - \varphi_0 K(\varepsilon)\eta) + \varepsilon^{-1/2}AN(\varepsilon)\times$$
$$\times(F(\varphi, \psi, \varepsilon) - 3\varphi_0 D^{-1}\varphi_0) + \varepsilon A F_1(\varphi, \psi, \varepsilon).$$

Equation (2.8) can easily be written in a more convenient integral form, if we transform the portion linear in η. For this purpose we must consider the differential equation $-w'' + \mu_0^2 w = 3wD^{-1}\varphi_0$. For $x > 0$ the function $\varphi_0(x)$ is its solution. With the aid of the well-known approach of reduction in order one can find a second solution $\varphi_1(x)$ such that $\varphi_1'(0) = 0$, $\varphi_1'\varphi_0 - \varphi_1\varphi_0' = 1$. Using the functions φ and φ_1 we construct the operator

$$Mg(x) = \varphi_0'(x)\int_0^x \varphi_1(t)\, D^{-1}\varphi_0(t)\, D^{-1}g(t)\, dt + \varphi_1'(x)\int_x^\infty \varphi_0(t)\, D^{-1}\varphi_0(t)\, D^{-1}g(t)\, dt.$$

As was shown in [6], functions of the form of Ah and $N(\varepsilon)h$ vanish at $x = 0$. Using the definition of Eq. (2.3) for the operator A, Eq. (2.8) can be written in the equivalent form

$$\eta = MF_2(\varphi_0, \varepsilon) + \varepsilon^{1/2} MF_3(\eta, \varphi_0, \varepsilon) \quad (x \geqslant 0). \tag{2.9}$$

It can easily be proved that φ_0, $D^{-1}\varphi_0 \sim \exp\{-\mu_0 x\}$, $\varphi_1 \sim \exp\{\mu_0 x\}$ as $x \to \infty$. Therefore, as follows from [6], the operator M acts from the space $E_{1/2}^+(\rho)$ at $\rho < \mu_0$ into the space $E_{1/2}^+(\rho)$ and is finite. From the estimates of the operators A, $N(\varepsilon)$ and $K(\varepsilon)$ performed in [6], and estimates of the complex function (2.5) we have the inequality uniform over ε $[0 < \varepsilon < \varepsilon_0(\mu_0, \rho)]$

$$\| MF_2(\varphi_0, \varepsilon)\|_{E_{1/2}^+(\rho)} \leqslant R < \infty.$$

Similarly it can be proved that for all η, η_1 from the sphere

$$B_{2R} = \left\{\eta \in \mathring{E}_{1/2}^+(\rho) \,|\, \|\eta - MF_2(\varphi_0, \varepsilon)\|_{E_{1/2}^+(\rho)} \leqslant 2R\right\}$$

the functions $MF_3(\eta, \varphi_0, \varepsilon)$ are uniformly limited over $\|MF_3(\eta, \varphi_0, \varepsilon)\|_{E_{1/2}^+(\rho)} \leq C_1(R)$, and the Lipshitz condition

$$\| MF_3(\eta, \varphi_0, \varepsilon) - MF_3(\eta_1, \varphi_0, \varepsilon)\|_{E_{1/2}^+(\rho)} \leqslant C_2(R)\|\eta - \eta_1\|_{E_{1/2}^+(\rho)}$$

is also satisfied. Therefore, for sufficiently small ε the transformation $\eta \to MF_2(\varphi_0, \varepsilon) + \varepsilon^{1/2}MF_3(\eta, \varphi_0, \varepsilon)$ acts from the sphere B_{2R} into itself and is compressive. According to the fixed point theorem, Eq. (2.9) has a unique solution. Therefore we have the following:

THEOREM. For sufficiently small ε the problem of Eqs. (1.2)-(1.5) has a nontrivial solution and

$$u(x, 1) \in \mathring{E}_{1/2}^+(\rho), \quad v(x, 1) \in E_{1/2}(\rho)$$

for $\rho < \mu_0$. The approximate solution is given by Eq. (2.7). In other words, to the accuracy of terms of order $\varepsilon^{1/2}O(1)$ the solution of the problem of Eqs. (1.2)-(1.5) at $y = 1$, $x = 0$ behaves like half a corresponding combined wave beginning from the point of symmetry.

Since the solution of the problem of Eqs. (1.2)-(1.5) belongs to the Holder space with exponent 1/2 and no higher, the form of the free surface on the physical plane is continually differentiable and its derivative belongs to the same Holder space. The free surface curvature has a singularity at the point of contact with the lid.

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